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A stochastic approach to Wilson loops

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Wilson loops $\exp(\mathrm{i} \oint A(x)\,\mathrm{d}x)$ are investigated in two-dimensional Euclidean space–time. The electromagnetic vector potential A is regarded as a generalized random field given by the stochastic partial differential equation $\partial A = F$ where ∂ is a first-order differential operator and F is white noise. We give a rigorous definition of Wilson loops and examine the properties of the N-loop Schwinger functions.

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1. Introduction

In analogy to the definition of the electromagnetic field tensor

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu ,$$

Albeverio and Høegh-Krohn suggested a model describing interacting quantum fields in Euclidean space–time [2,3]. They consider a stochastic partial differential equation of the form

$$\partial A = F ,$$

where ∂ is the first-order differential operator

$$\partial = \frac{\partial}{\partial x_1} e_1 - \sum_{k=2}^{d} \frac{\partial}{\partial x_k} e_k$$

and $\{e_1, \ldots, e_d\}$ is the standard basis of $\mathbb{R}^d$. The fields A and F are no longer vector fields, but multicomponent generalized random fields.

Albeverio et al. assume that the generalized random field F is white noise, in general non-Gaussian, because they want A to have the Markov property. If in addition A is reflection invariant, one can try to prove Osterwalder–Schrader positivity, at least on some subspace of the test function space, see, e.g., ref. [15] for the two-dimensional case.

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The equation $\partial A = F$ only makes sense if there is also a multiplication $\mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^d$ so that d must be in $\{1,2,4,8\}$. A general overview of this model can be found in ref. [7].

In the four-dimensional case we have the noncommutative field of quaternions. If F is Gaussian white noise, A is the free electromagnetic field in the Feynman gauge, whereas the non-Gaussian case corresponds to some interaction. The case $d = 4$ is treated in refs. [2,3,5,7]. Osipov has investigated the octonionic case, see the references in ref. [15].

If $d = 2$ we have the field of complex numbers. The two-dimensional case has been studied in refs. [6,8,9,15]. There is a connection to Yang–Mills theory, see ref. [12].

In this article, which is devoted to the two-dimensional case, we point out that Wilson loops are stochastic cosurfaces in the sense of ref. [4]. We plan to generalize our results to the case of manifolds.

2. Construction of the generalized random field A

Let us first introduce the generalized random field F. A generalized random field is a continuous linear map from some test function space T, equipped with a topology, into the random variables on a fixed probability space $(\Omega, \mathcal{A}, P)$, i.e.,

$$F : T \longrightarrow \{\mathbb{R}\text{-valued random variables on } (\Omega, \mathcal{A}, P)\}$$

such that $\forall \lambda_1, \lambda_2 \in \mathbb{R}, f_1, f_2 \in T$,

$$F(\lambda_1 f_1 + \lambda_2 f_2) = \lambda_1 F(f_1) + \lambda_2 F(f_2) \quad \text{almost surely}$$

and

$$f_n \overset{n\to\infty}{\longrightarrow} f \quad \Rightarrow \quad F(f_n) \overset{n\to\infty}{\longrightarrow} F(f)\,.$$

$F(f_n) \overset{n\to\infty}{\longrightarrow} F(f)$ holds, for example, in probability.

We always assume that the test function space T is a vector space over $\mathbb{R}$. If T is a space of functions $\mathbb{R}^n \to \mathbb{R}$, F is a scalar generalized random field whereas in the case of functions $\mathbb{R}^n \to \mathbb{R}^m$, F is a multicomponent generalized random field. On the formal level we have

$$F(f) = \int_{\mathbb{R}^n} \sum_{j=1}^{m} X_j f_j(x)\, \mathrm{d}^n x\,,$$

where $f : \mathbb{R}^n \to \mathbb{R}^m$ and X_j, $j = 1, \ldots, m$, are the components of a random field, i.e. a stochastic process indexed by $\mathbb{R}^m$.

Let T' denote the topological dual of T and let $(\,\cdot\,,\,\cdot\,)$ denote the canonical pairing between T' and T. There is, roughly speaking, a one-to-one correspondence between generalized random fields indexed by T and probability measures on T'. Given a probability measure on T', $f \mapsto (\,\cdot\,, f)$ is a generalized random field. Conversely, if we can apply Minlos' theorem (cf. ref. [13]), the characteristic functional of the generalized random field F, $f \mapsto E(e^{iF(f)})$, defines a probability measure on T'.

Now we introduce F via its characteristic functional

$$\Phi_F(f) = E(e^{iF(f)}) = \exp\Big(\int_{\mathbb{R}^2} h(f(x))\, d^2x\Big), \tag{1}$$

where $h : \mathbb{R}^2 \to \mathbb{C}$ is of the form

$$h(x) = \int_{\mathbb{R}^2\setminus\{0\}} \Big(e^{i\langle\alpha,x\rangle} - 1 - i\langle\alpha, x\rangle\Big) d\nu(\alpha) - \langle x, Mx\rangle. \tag{2}$$

The form (2) is the so-called Kolmogorov canonical representation (cf. refs. [9,14]). M is a positive definite 2×2-matrix. The measure ν, the Lévy measure, is assumed to have finite second moments: $\int_{\mathbb{R}^2\setminus\{0\}} |\alpha|^2\, d\nu(\alpha) < \infty$. In the purely Gaussian case we have $\nu = 0$.

Note that the special form of the characteristic functional implies that F is independent at every point, i.e., if $f_1 \cdot f_2 \equiv 0$ then $F(f_1)$ and $F(f_2)$ are independent. As a consequence of the Kolmogorov canonical representation (2) $-h$ is a continuous negative definite function [10] and the following inequality holds:

$$|h(x)| \le M \cdot |x|^2 \qquad \forall x \in \mathbb{R}^2, \tag{3}$$

where M is a suitable constant, see ref. [9].

We assume that F is indexed by $\mathcal{S}(\mathbb{R}^2, \mathbb{C})$, the space of rapidly decreasing functions $\mathbb{R}^2 \to \mathbb{C}$. Applying Minlos' theorem, one can show that there is a unique probability measure μ_F on the dual space $\mathcal{S}'(\mathbb{R}^2, \mathbb{C})$ such that

$$\Phi_F(f) = \int_{\mathcal{S}'} e^{i(\xi,f)}\, d\mu_F(\xi). \tag{4}$$

Let us now return to the equation $\partial A = F$. Apart from the operator

$$\partial = \partial/\partial x_1 - i\partial/\partial x_2,$$

we also consider the operator

$$\overline{\partial} = \partial/\partial x_1 + i\partial/\partial x_2.$$

One easily proves that $i\overline{\partial}$ is the exterior derivative and that $-\overline{\partial}$ is the formal adjoint of ∂. $\frac{1}{2}\overline{\partial} = \partial/\partial\bar{z}$ is the Cauchy–Riemann operator.

Given a function $f \in \mathcal{S}(\mathbb{R}^2, \mathbb{C})$, the equation $\partial A = F$ reads

$$\partial A(f) = A(-\overline{\partial} f) = F(f),$$

so that

$$A(f) = A((-\overline{\partial})(-\overline{\partial})^{-1} f) = F((-\overline{\partial})^{-1} f). \tag{5}$$

The fundamental solution of $-\bar{\partial}$ can be constructed by using the fundamental solution of the two-dimensional Laplacian and the identity

$$\partial\bar{\partial} = \bar{\partial}\partial = \Delta .$$

We have

$$\delta = \Delta\Big(\frac{1}{2\pi} \ln|x| \Big) = -\bar{\partial}\Big(-\partial\frac{1}{2\pi} \ln|x| \Big) = -\bar{\partial}\Big(-\frac{1}{2\pi x} \Big).$$

It can be shown that the above equality does not only hold in the sense of $\mathcal{D}'$, but even in the sense of $\mathcal{S}'$. We denote the fundamental solution of $-\bar{\partial}$ by $S(x) := -1/2\pi x$. Using Fourier transform methods, we get the following proposition (cf. ref. [9]).

Proposition 2.1.

$$f \in \mathcal{S}(\mathbb{R}^2, \mathbb{C}) \quad \Rightarrow \quad S*f \in L^p(\mathbb{R}^2, \mathbb{C}) \;\; \forall p \in \,]2, \infty] .$$

Put

$$\mathcal{S}_0 := \mathcal{S}_0(\mathbb{R}^2, \mathbb{C}) := \{f \in \mathcal{S}(\mathbb{R}^2, \mathbb{C}) \,|\, \textstyle\int_{\mathbb{R}^2} f(x)\, \mathrm{d}^2x = 0\}$$

and let $f \in \mathcal{S}(\mathbb{R}^2, \mathbb{C})$. *We have* $f \in \mathcal{S}_0(\mathbb{R}^2, \mathbb{C}) \iff S*f \in L^2(\mathbb{R}^2, \mathbb{C})$. □

Because of (5) the characteristic functional of the field A must be

$$\Phi_A(f) = E(\mathrm{e}^{\mathrm{i}A(f)}) = \exp\Big(\int_{\mathbb{R}^2} h(S*f(x))\, \mathrm{d}^2x \Big). \tag{6}$$

We assume that A is indexed by $\mathcal{S}_0$ because the condition $\int_{\mathbb{R}^2} f(x)\, \mathrm{d}^2x = 0$ ensures that $\int_{\mathbb{R}^2} h(S*f(x))\, \mathrm{d}^2x$ exists, cf. inequality (3) and proposition 2.1. Osipov [15] uses a different test function space,

$$\mathcal{S}_{0,T} = \{f \in \mathcal{S}_0(\mathbb{R}^2, \mathbb{C}) \mid \partial f_1/\partial x_1 + \partial f_2/\partial x_2 = 0\},$$

to prove Osterwalder–Schrader positivity for the field A. Since the functions f_n^C, which we introduce in section 3, are in $\mathcal{S}_{0,T}$ we could also use $\mathcal{S}_{0,T}$ instead of $\mathcal{S}_0$.

Assuming that A is indexed by $\mathcal{S}_0$, Minlos' theorem yields that there is a unique probability measure μ_A on $\mathcal{S}_0'$ such that

$$\Phi_A(f) = \int_{\mathcal{S}_0'} \mathrm{e}^{\mathrm{i}(\xi, f)}\, \mathrm{d}\mu_A(\xi). \tag{7}$$

Note that the random variables $F(f_1)$, $f_1 \in \mathcal{S}(\mathbb{R}^2, \mathbb{C})$, and $A(f_2), f_2 \in \mathcal{S}_0(\mathbb{R}^2, \mathbb{C})$, are defined on different probability spaces.

Since in the sequel we want to write down expressions like $F(1_B)$, where 1_B is the indicator function of a Borel set B, we have to extend the operator F to a larger space. It is well known that Gaussian white noise can be extended to L^2, cf. ref. [13]. The following lemma shows that such an extension can also be constructed in the non-Gaussian case, provided one has a Kolmogorov canonical representation (2).

Lemma 2.2.
$$E(F(f)^2) \leq 2M \cdot \|f\|_2^2 \qquad \forall f \in \mathcal{S}(\mathbb{R}^2, \mathbb{C}),$$
where M is the constant in (3). □

Let $L^2(\mathcal{S}', \mu_F)$ denote the space of random variables on $\mathcal{S}'$ that are square-integrable with respect to μ_F. Lemma 2.2 shows that $F : \mathcal{S}(\mathbb{R}^2, \mathbb{C}) \to L^2(\mathcal{S}', \mu_F)$ can be uniquely extended to a continuous operator $L^2(\mathbb{R}^2, \mathbb{C}) \to L^2(\mathcal{S}', \mu_F)$, which for notational convenience we shall also denote by F.

It is easy to prove that the characteristic functional of F is given by (1) and (2) $\forall f \in L^2(\mathbb{R}^2, \mathbb{C})$ and that F is still independent at every point, i.e., $\forall f_1, f_2 \in L^2(\mathbb{R}^2, \mathbb{C})$ with the property $f_1 \cdot f_2 \equiv 0$ the random variables $F(f_1)$ and $F(f_2)$ are independent. We remark that this L^2-extension also works if we have a generalized random field that has a characteristic functional of the form (1) and (2) and that is indexed by test functions $\mathbb{R}^n \to \mathbb{R}^m$.

As a consequence of this L^2-extension we can put $A(f) = F(S*f)$ because $\forall f \in \mathcal{S}_0$ we have $S*f \in L^2$.

The following theorem summarizes the results of this section.

Theorem 2.3. *Let $f_1 \in \mathcal{S}_0(\mathbb{R}^2, \mathbb{C})$ and $f_2 \in \mathcal{S}(\mathbb{R}^2, \mathbb{C})$ be test functions. We have two probability spaces $(\mathcal{S}_0', \mathcal{A}_0, \mu_A)$ and $(\mathcal{S}', \mathcal{A}, \mu_F)$ and can look upon the equation $\partial A = F$ in two different ways:*

(i) If we put $A(f_1) = F(S*f_1)$ the random variables $A(f_1)$ and $F(f_2)$ are defined on the same probability space $(\mathcal{S}', \mathcal{A}, \mu_F)$ and $\partial A(f_1) = F(f_1)$ holds almost surely.

(ii) If we regard $A(f_1)$ as a random variable on $(\mathcal{S}_0', \mathcal{A}_0, \mu_A)$, $A(f_1)$ and $F(f_2)$ are defined on different probability spaces and the equation $\partial A(f_1) = F(f_1)$ holds in law.

The random variables $A(f_1) : \mathcal{S}_0' \to \mathbb{R}$ and $A(f_1) : \mathcal{S}' \to \mathbb{R}$ are equal in law. □

3. Wilson loops

On the formal level, Wilson loops are $\exp(\mathrm{i} \oint_C A(x)\, \mathrm{d}x)$. Application of Stokes' theorem yields $\oint_C A(x)\, \mathrm{d}x = \int_B F(x)\, \mathrm{d}^2x$, where B is the interior of the curve C. However, this has to be interpreted carefully: $A(x)$ is just a formal expression since A is a generalized random field. Our idea is to construct a sequence of test functions that converges to the delta distribution on the curve. Tamura [17] has carried out a similar construction in the four-dimensional case.

We assume that the curve C is closed, has only finitely many self-intersections and that it can be parametrized by a map that is piecewise C^1. In section 5 we consider the more general case of curves that are not necessarily closed.

We take a function $\varphi \in C_0^\infty(\mathbb{R}^2, \mathbb{R})$ with the properties $\varphi \geq 0$, $\int_{\mathbb{R}^2} \varphi(x)\, \mathrm{d}^2x = 1$ and $\operatorname{supp} \varphi \subseteq [-1,1] \times [-1,1]$. Let $\varphi_n(x) := n^2 \varphi(nx)$ and

$$f_n^C(x) := \oint_C \varphi_n(x-z)\, \mathrm{d}z \,. \tag{8}$$

It is easily seen that $\forall n \in \mathbb{N}$ $f_n^C \in C_0^\infty(\mathbb{R}^2, \mathbb{C})$ and that $\int_{\mathbb{R}^2} f_n^C(x)\, \mathrm{d}^2x = 0$, i.e., $f_n^C \in \mathcal{S}_0$. We have $f_n^C \overset{n\to\infty}{\longrightarrow} \delta_C$ in the sense of $\mathcal{S}'$, where δ_C is the delta distribution on the curve C.

Let

$$n_C(x) := \frac{1}{2\pi \mathrm{i}} \oint_C \frac{1}{z-x}\, \mathrm{d}z$$

denote the winding number. We can prove the pointwise convergence

$$(S*f_n^C)(x) \overset{n\to\infty}{\longrightarrow} (S*\delta_C)(x) = \oint_C S(x-z)\, \mathrm{d}z = \mathrm{i}\, n_C(x) \quad \forall x \notin C \,. \tag{9}$$

Theorem 3.1.

We assume that the function h in eq. (2) satisfies the condition

$$|h(0,x_2)| \leq K \cdot |x_2|^\zeta \qquad \forall x_2 \in \mathbb{R}, \tag{10}$$

where $K \geq 0$ and $\zeta \in [0,1[$.

– If we regard $A(f_n^C)$, $n \in \mathbb{N}$, as random variables on $(\mathcal{S}', \mathcal{A}, \mu_F)$ then $\mathrm{e}^{\mathrm{i}A(f_n^C)} \overset{n\to\infty}{\longrightarrow} \mathrm{e}^{\mathrm{i}F(\mathrm{i}n_C)}$ in the sense of $L^p(\mathcal{S}', \mu_F)$ $\forall p \in [1, \infty[$.

– If we regard $A(f_n^C)$, $n \in \mathbb{N}$, as random variables on $(\mathcal{S}_0', \mathcal{A}_0, \mu_A)$ then $\mathrm{e}^{\mathrm{i}A(f_n^C)}$, $n \in \mathbb{N}$, is a Cauchy sequence in $L^p(\mathcal{S}_0', \mu_A)$ $\forall p \in [1, \infty[$.

– The sequence $A(f_n^C)$, $n \in \mathbb{N}$, converges weakly to $F(\mathrm{i}n_C)$.

Note that in the last assertion we do not have to specify the probability space on which the $A(f_n^C)$, $n \in \mathbb{N}$, are defined.

Sketch of the proof. (Details can be found in ref. [9].) Let us first regard $A(f_n^C), n \in \mathbb{N}$, as random variables on $(\mathcal{S}', \mathcal{A}, \mu_F)$. It is sufficient to consider the case $p = 2$.

$$\|\mathrm{e}^{\mathrm{i}A(f_n^C)} - \mathrm{e}^{\mathrm{i}F(\mathrm{i}n_C)}\|_2^2$$
$$= \int_{\mathcal{S}'} \Big(2 - \exp[\mathrm{i}F(S*f_n^C - \mathrm{i}n_C)] - \exp[-\mathrm{i}F(S*f_n^C - \mathrm{i}n_C)]\Big) \mathrm{d}\mu_F \,.$$

If

$$\lim_{n\to\infty} \int_{\mathbb{R}^2} h(S*f(x) - \mathrm{i}n_C(x))\, \mathrm{d}^2x = 0 \tag{11}$$

we have

$$\lim_{n\to\infty} \int_{\mathcal{S}'} e^{iF(S * f_n^C - in_C)}\, d\mu_F = 1\,,$$

which proves our first assertion.

We use condition (10) to interchange limit and integral in (11). By (9), $h(0) = 0$ and the continuity of h we get (11).

Let us now prove that $e^{iA(f_n^C)}, n \in \mathbb{N}$, is a Cauchy sequence in $L^p(\mathcal{S}_0', \mu_A)$. Again it is sufficient to consider the case $p = 2$. Employing that $e^{iA(f_n^C)}, n \in \mathbb{N}$, is Cauchy in $L^2(\mathcal{S}', \mu_F)$ and that $\forall f \in \mathcal{S}_0\ A(f) : \mathcal{S}_0' \to \mathbb{R}$ and $A(f) : \mathcal{S}' \to \mathbb{R}$ are equal in law, we get for sufficiently large n, m,

$$\begin{aligned} \varepsilon > \left\| e^{iA(f_n^C)} - e^{iA(f_m^C)} \right\|_{2,\mathcal{S}'}^2 &= \int_{\mathcal{S}'} \left(2 - e^{iA(f_n^C - f_m^C)} - e^{iA(f_m^C - f_n^C)}\right) d\mu_F \\ &= \left\| e^{iA(f_n^C)} - e^{iA(f_m^C)} \right\|_{2,\mathcal{S}_0'}^2 . \end{aligned}$$

We still have to prove that $A(f_n^C) \overset{n\to\infty}{\longrightarrow} F(in_C)$ weakly. Using (9) and the continuity of h, we see the convergence of the characteristic functions

$$\begin{aligned} \lim_{n\to\infty} E\left(e^{itA(f_n^C)}\right) &= \lim_{n\to\infty} \exp\left(\int_{\mathbb{R}^2} h(t\, S * f_n^C(x))\, d^2x \right) \\ &= \exp\left(\int_{\mathbb{R}^2} h(\lim_{n\to\infty} t\, S * f_n^C(x))\, d^2x \right) \\ &= \exp\left(\int_{\mathbb{R}^2} h(t\, i\, n_C(x))\, d^2x \right) = E\left(e^{itF(in_C)}\right), \end{aligned}$$

where we employed condition (10) to interchange limit and integral. □

It is reasonable to expect that $\oint_C A(x)\, dx$ is gauge invariant.

If χ is a generalized random field having the property $\Delta\chi = \partial\bar{\partial}\chi = 0$, we see that $A + \bar{\partial}\chi$ is also a solution of $\partial A = F$. We assume that χ is indexed by a test function space T having the property $f \in \mathcal{S}_0 \Rightarrow \partial f \in T$ and denote the corresponding probability space by $(T', \mathcal{A}_\chi, \mu_\chi)$. Under these assumptions $(A + \bar{\partial}\chi)(f) = A(f) + \chi(-\partial f)$ is a random variable defined on $(\mathcal{S}_0, \mathcal{A}_0, \mu_A) \otimes (T', \mathcal{A}_\chi, \mu_\chi)$.

The Wilson loop is gauge invariant in the sense that

$$\exp\left(i \oint_C (A + \bar{\partial}\chi)(x)\, dx \right) = \exp\left(i \oint_C A(x)\, dx \right) \qquad \text{almost surely.} \qquad (12)$$

$\oint_C A(x)\, dx$, regarded as weak limit of $A(f_n^C), n \in \mathbb{N}$, is gauge invariant in the sense that

$$\oint_C (A + \overline{\partial}\chi)(x)\,\mathrm{d}x = \oint_C A(x)\,\mathrm{d}x \qquad \text{in law.} \tag{13}$$

Note that condition (10) excludes the Gaussian case $h(x) = -\frac{1}{2}|x|^2$. We just give one example of a function h that has a Kolmogorov canonical representation (2) and satisfies (10). Other examples can be found in ref. [9].

Example 3.2. The function $h(x) := \exp(-\frac{1}{2}|x|^2) - 1$ fulfills condition (10) and has the representation

$$\begin{aligned}\exp(-\tfrac{1}{2}|x|^2) - 1 &= \int_{\mathbb{R}^2\setminus\{0\}} \left(e^{i\langle\alpha,x\rangle} - 1\right) \mathrm{d}\nu(\alpha) \\ &= \int_{\mathbb{R}^2\setminus\{0\}} \left(e^{i\langle\alpha,x\rangle} - 1 - i\langle\alpha,x\rangle\right) \mathrm{d}\nu(\alpha),\end{aligned}$$

where ν is the bivariate normal distribution.

Let us see how we can get rid of the restrictive condition (10). This condition ensures that we can interchange limit and integral in the proof of theorem 3.1. If the function h does not satisfy (10), we can regard the proof of theorem 3.1 as a formal calculation that motivates the following definition.

Definition 3.3. We assume that the curve C has only finitely many self-intersections and that it can be parametrized by a map that is piecewise C^1. Let n_C denote the winding number. We define the Wilson loop by

$$\oint_C A(x)\,\mathrm{d}x := F(i\,n_C). \tag{14}$$

We remark that the preceding definition depends on the fact that $S * \delta_C = i\,n_C \in L^2$, which is specific for the two-dimensional case.

Let us now have a look at the distribution of $\oint_C A(x)\mathrm{d}x$ and $\exp(i\oint_C A(x)\mathrm{d}x)$. Since we assume that the curve C has only finitely many self-intersections, the set $\{n_C \neq 0\}$ has only finitely many connected components denoted by D_l, $l = 1, \ldots, m$. We have $n_C(x) = \sum_{l=1}^m n_l 1_{D_l}(x)$ because the winding number is constant on each D_l. Let P_C and Q_C denote the distributions of $\oint_C A(x)\,\mathrm{d}x$ and $\exp(i\oint_C A(x)\,\mathrm{d}x)$, respectively. Furthermore let λ^2 denote the Lebesgue measure on $\mathbb{R}^2$. P_C is given by the characteristic function

$$t \longmapsto E\left(\exp\left(i\,t\oint_C A(x)\,\mathrm{d}x\right)\right) = \exp\left(\sum_{l=1}^m h(i\,n_l\,t)\,\lambda^2(D_l)\right). \tag{15}$$

If C is a simple closed curve and $n_C = +1$ on the interior B we have the "area law"

$$E\left(\exp\left(\mathrm{i}t\oint_C A(x)\,\mathrm{d}x\right)\right) = \exp[h(\mathrm{i}t)\lambda^2(B)]\,. \tag{16}$$

The characteristic function $(x_1,x_2) \mapsto \mathrm{e}^{h(x_1,x_2)}$ defines a probability measure P on $\mathbb{R}^2$. Let Π_s denote the projection map $(x_1,x_2) \mapsto sx_2$, where $s \in \mathbb{R}$. Since $\Pi_s(P)$ is infinitely divisible there is a unique continuous convolution semigroup $P_{s,r}$, $r \geq 0$, with the property $P_{s,1} = \Pi_s(P)$. P_C is infinitely divisible since

$$P_C = \mathop{\bigstar}_{l=1}^{m} P_{n_l,\lambda^2(D_l)}\,. \tag{17}$$

Let $g : \mathbb{R} \to \mathrm{S}^1$, $x \mapsto \mathrm{e}^{\mathrm{i}x}$. $Q_{s,r} = g(P_{s,r})$, $r \geq 0$, is a family of continuous convolution semigroups on S^1. Q_C is an infinitely divisible probability measure on S^1 which is given by

$$Q_C = \mathop{\bigstar}_{l=1}^{m} Q_{n_l,\lambda^2(D_l)}\,. \tag{18}$$

Let us make a short remark on the static potential. If C is a rectangle with sides of length R and T, the static potential is defined by

$$V(R) := -\lim_{T\to\infty} \frac{1}{T} \ln E\left(\exp\left(\mathrm{i}\oint_C A(x)\,\mathrm{d}x\right)\right). \tag{19}$$

We can immediately evaluate (19) by (16): $V(R) = -h(\mathrm{i})R$. If the Lévy measure ν is invariant under the reflection $x \mapsto -x$ we have $h(x) \in \mathbb{R}$ and $-h(x) \geq 0$ $\forall x \in \mathbb{R}^2$. Assuming that $h(\mathrm{i}) \neq 0$, we have $\lim_{R\to+\infty} V(R) = +\infty$, which is important in elementary particle physics. We refer the reader to ref. [17] and the references quoted there.

4. Properties of N-loop Schwinger functions

The gauge invariant expectation values

$$S_N(C_1,\ldots,C_N) = E\left(\prod_{j=1}^{N} \exp\left(\mathrm{i}\oint_{C_j} A(x)\,\mathrm{d}x\right)\right) \tag{20}$$

are called N-loop Schwinger functions. We have $\forall t_1,\ldots,t_N \in \mathbb{R}$

$$E\left(\prod_{j=1}^{N} \exp\left(\mathrm{i}t_j\oint_{C_j} A(x)\,\mathrm{d}x\right)\right) = \exp\left(\int_{\mathbb{R}^2} h\left(\mathrm{i}\sum_{j=1}^{N} t_j n_{C_j}(x)\right)\mathrm{d}^2x\right) \tag{21}$$

so that

$$S_N(C_1,\dots,C_N) = E\Big(\prod_{j=1}^{N}\exp\Big(\mathrm{i}\oint_{C_j} A(x)\,\mathrm{d}x\Big)\Big)$$
$$= \exp\Big(\int_{\mathbb{R}^2} h\Big(\mathrm{i}\sum_{j=1}^{N} n_{C_j}(x)\Big)\,\mathrm{d}^2x\Big). \tag{22}$$

Let $U := \{x \in \mathbb{R}^2\setminus C \mid n_C(x) \neq 0\}$ denote the set spanned by the curve C. If the sets $U_1,\dots,U_N$ spanned by the curves $C_1,\dots,C_N$ are mutually disjoint, the random variables $\oint_{C_j} A(x)\,\mathrm{d}x$, $j \in \{1,\dots,N\}$, are independent and

$$S_N(C_1,\dots,C_N) = E\Big(\prod_{j=1}^{N}\exp\Big(\mathrm{i}\oint_{C_j} A(x)\,\mathrm{d}x\Big)\Big)$$
$$= \prod_{j=1}^{N} E\Big(\exp\Big(\mathrm{i}\oint_{C_j} A(x)\,\mathrm{d}x\Big)\Big). \tag{23}$$

Our next aim is to verify that the N-loop Schwinger functions satisfy the conditions formulated by Fröhlich, Osterwalder, and Seiler [16]. We shall call these conditions Fröhlich–Osterwalder–Seiler axioms. In order to formulate these axioms, we have to equip the set of all curves with a topology. We distinguish between the notion of a path $\gamma : [a,b] \to \mathbb{R}^2$ and the notion of a curve, an oriented set $C \subseteq \mathbb{R}^2$ with $\gamma([a,b]) = C$. Let $\mathcal{C}$ denote the set of all closed curves that are piecewise differentiable and that have only finitely many self-intersections. It can be proved that

$$d(C_1,C_2) := \inf_{\gamma_1,\gamma_2} \|\gamma_1 - \gamma_2\|_\infty \tag{24}$$

is a metric on $\mathcal{C}$. The infimum is taken over all paths γ_1, γ_2 that parametrize C_1 and C_2, respectively. From now on we assume that $\mathcal{C}$ is equipped with the topology induced by (24).

We remark that whenever $C_k \overset{k\to\infty}{\longrightarrow} C$ the winding numbers converge pointwise, $n_{C_k}(x) \overset{k\to\infty}{\longrightarrow} n_C(x)\ \ \forall x \in \mathbb{R}^2\setminus\{C \cup \bigcup_{k=1}^{\infty} C_k\}$, and in the sense of $L^2(\mathbb{R}^2)$ as well.

Employing lemma 2.2, we can prove that

$$(C_1,\dots,C_N) \longmapsto \sum_{j=1}^{N} \oint_{C_j} A(x)\,\mathrm{d}x \tag{25}$$

is continuous in the sense of $L^p(\mathcal{S}',\mu_F)\ \ \forall p \in [1,2]$.

$$(C_1,\dots,C_N) \longmapsto \prod_{j=1}^{N}\exp\Big(\mathrm{i}\oint_{C_j} A(x)\,\mathrm{d}x\Big) \tag{26}$$

is continuous in the sense of $L^p(\mathcal{S}', \mu_F)$ $\forall p \in [1,\infty[$ and

$$(C_1,\ldots,C_N) \longmapsto E\Big(\prod_{j=1}^{N} \exp\Big(\mathrm{i} \oint_{C_j} A(x)\,\mathrm{d}x\Big)\Big) \tag{27}$$

is continuous.

Let us point out that Wilson loops can be regarded as noise in the sense of ref. [1].

We consider a set M and a ring $\mathcal{F} \subseteq \mathcal{P}(M)$, i.e., $\emptyset \in \mathcal{F}$ and if A and B are in $\mathcal{F}$ then $A \cup B$ and $A \setminus B$ are in $\mathcal{F}$, too. Furthermore we have a probability space $(\Omega, \mathcal{A}, P)$ and a monoid G with unit element e.

A map $\xi : \mathcal{F} \longrightarrow \{ G\text{-valued random variables on } (\Omega, \mathcal{A}, P) \}$ is called noise if

$$A, B \in \mathcal{F},\ A \cap B = \emptyset \Rightarrow \xi(A) \text{ and } \xi(B) \text{ are independent}$$
$$\text{and } \xi(A \cup B) \stackrel{\mathrm{d}}{=} \xi(A)\cdot\xi(B),$$
$$\xi(\emptyset) \stackrel{\mathrm{d}}{=} e,$$

where $\stackrel{\mathrm{d}}{=}$ denotes equality in law. ξ is called continuous noise if additionally

$$A_n \in \mathcal{F},\ \ A_n \downarrow \emptyset \quad \Rightarrow \quad \xi(A_n) \longrightarrow \xi(\emptyset).$$

Let us take the ring $\mathcal{F}$ generated by the bounded simply connected Borel subsets of $\mathbb{R}^2$. Note that each element of $\mathcal{F}$ is a bounded Borel set. The map $B \mapsto F(\mathrm{i}\,1_B)$ is an $\mathbb{R}$-valued noise on $\mathbb{R}^2$, continuous in the sense of $L^p(\mathcal{S}', \mu_F)$ $\forall p \in [1,2]$. The S^1-valued noise $B \mapsto \mathrm{e}^{\mathrm{i}F(\mathrm{i}\,1_B)}$ is continuous in the sense of $L^p(\mathcal{S}', \mu_F)$ $\forall p \in [1,\infty[$.

Theorem 4.1 (Verification of the axioms). *The N-loop Schwinger functions $S_N(C_1,\ldots,C_N)$, $N \in \mathbb{N}$, satisfy all Fröhlich–Osterwalder–Seiler axioms (cf. ref. [16], pp. 164–167).*

(FOS 0): Technical assumption. *$S_N : \mathcal{C}^N \longrightarrow \mathbb{C}$ is continuous.*
We remark that some other technical assumptions are mentioned in ref. [16] which are trivial in our case.

(FOS 1): Symmetry. *$S_N(C_{\sigma(1)},\ldots,C_{\sigma(N)}) = S_N(C_1,\ldots,C_N)$ for all permutations σ.*

(FOS 2): Euclidean invariance.

$$S_N(TC_1,\ldots,TC_N) = S_N(C_1,\ldots,C_N) \quad \forall T \in \mathbb{R}^2 \odot \mathrm{SO}(2),$$

where $\mathbb{R}^2 \odot \mathrm{SO}(2)$ is the semidirect product of $\mathbb{R}^2$ and $\mathrm{SO}(2)$. If the function h in (2) is invariant under the reflection $(0,x) \mapsto (0,-x)$, S_N is even invariant under $\mathbb{R}^2 \odot \mathrm{O}(2)$.

(FOS 3): Osterwalder–Schrader positivity. *We identify the first component and the time. Let V be the complex vector space generated by random variables of the* *form* $\prod_{j=1}^{N} \exp(\mathrm{i} \oint_{C_j} A(x)\,\mathrm{d}x)$, *where we assume that the sets* U_j *spanned by the curves* C_j *are mutually disjoint. Note that* N *is not fixed. Let* V_+ *denote the subspace spanned by* $\prod_{j=1}^{N} \exp(\mathrm{i} \oint_{C_j} A(x)\,\mathrm{d}x)$, *where the sets* U_j *are mutually disjoint and are in the right half-space* $\mathbb{R}_+ \times \mathbb{R}$. *We define* V_- *correspondingly. Let* R *be the antilinear map* $V_+ \to V_-$ *reflecting the curves at* $t = 0$ *and taking complex conjugates of the coefficients. Then the following Osterwalder–Schrader positivity condition holds:*

$$E\big(P \cdot R(P)\big) \geq 0 \qquad \forall P \in V_+ .$$

(FOS 4): Clustering. *If* $P \in V$, *let* P^a *denote the random variable obtained by translating all curves by a vector* $a \in \mathbb{R}^2$. *Thus we have a linear map* $V \to V$, $P \mapsto P^a$. *The following clustering condition holds:*

$$\lim_{|a| \to +\infty} E(Q \cdot P^a) = E(Q) \cdot E(P) \qquad \forall Q, P \in V .$$

Proof. (FOS 0) is (27). (FOS 1) is trivial. (21) and the invariance properties of the Lebesgue measure yield (FOS 2). $\forall P \in V_+$ the random variables P and $R(P)$ are independent and $E(R(P)) = \overline{E(P)}$, which proves (FOS 3). (FOS 4) follows from the fact that for sufficiently large $|x|$, Q and P are independent.□

$\langle P, Q\rangle := E(P \cdot R(Q))$ obviously is a sesquilinear form on V_+. Employing the Osterwalder–Schrader positivity condition we see that $\langle \cdot, \cdot \rangle$ is positive definite on the physical Hilbert space $H := V_+/N$, where $N := \{P \in V_+ \mid \langle P, P\rangle = 0\}$.

Let $T_t : P \mapsto P^{(t,0)}$, where $t \geq 0$, i.e., we translate all curves in positive time direction. T_t, $t \geq 0$, is the semigroup of translations in time.

It is obvious that the physical Hilbert space is at least one-dimensional. The following theorem shows that H is exactly one-dimensional.

Theorem 4.2. *The physical Hilbert space* H *is a one-dimensional complex vector space. Each element of the semigroup of translations in time is the identity on* H.

Proof. We show that any two vectors $P, Q \in H$ are linearly dependent. Let $Q \neq 0$ and let $\lambda \in \mathbb{C}$. If we put $\lambda = -E(P)/E(Q)$ we have

$$\langle P + \lambda Q, P + \lambda Q\rangle = (E(P) + \lambda E(Q)) \cdot \overline{(E(P) + \lambda E(Q))} = 0 .$$

We conclude that H is one-dimensional. Let us fix $Q \neq 0$. We have $\forall P \in H$, $\forall t \geq 0$

$$\langle T_t P, Q\rangle = S_1(T_t P) \cdot \overline{S_1(Q)} = S_1(P) \cdot \overline{S_1(Q)} = \langle P, Q\rangle ,$$

where we employed the invariance property (FOS 2). It follows that $T_t = \mathrm{id}_H \quad \forall t \geq 0$. □

We mention that Wilson loops on a two-dimensional lattice were studied by Dosch and Müller [11], who proved a factorization lemma analogous to (23). Because of this factorization lemma the corresponding lattice Schwinger functions also satisfy the Fröhlich–Osterwalder–Seiler axioms and we have a one-dimensional Hilbert space, too. The condition of Euclidean invariance has of course to be dropped because this is a lattice theory.

5. Wilson loops regarded as stochastic cosurfaces

In this section we shall point out the connection between Wilson loops and stochastic cosurfaces. The notion of stochastic cosurfaces was developed by Albeverio and Høegh-Krohn in 1984, cf. ref. [4].

Let us assume that we have a two-dimensional random field $X_z, z \in \mathbb{C}$, i.e., a stochastic process indexed by $\mathbb{C} \cong \mathbb{R}^2$. Proceeding formally, we consider the map $K : C \mapsto \int_C X_z \, dz$, which has the following properties:

(i) Let $-C$ denote the curve that equals C as a point set, but with opposite orientation. Then we have almost surely $K(-C) = -K(C)$.

(ii) If we have two curves C_1 and C_2 such that the terminal point of C_1 is the initial point of C_2, we have almost surely $K(C_1 \cup C_2) = K(C_1) + K(C_2)$.

The preceding considerations motivate the following definition.

Definition 5.1. Let $\mathcal{C}$ be a set of curves in $\mathbb{R}^2$ and let G be a group.

A stochastic cosurface is a map

$$K : \mathcal{C} \longrightarrow \{ G\text{-valued random variables on } (\Omega, \mathcal{A}, P) \}$$

with the properties that almost surely $K(-C) = K(C)^{-1}$ and $K(C_1 \cup C_2) = K(C_1) \cdot K(C_2)$ whenever the terminal point of C_1 equals the initial point of C_2.

It is often useful to assume that G is compact in order to have a finite Haar measure on G.

We remark that the notion of stochastic cosurfaces can be generalized: $\mathbb{R}^2$ can be replaced by an oriented Riemannian manifold M of dimension d and $\mathcal{C}$ can be replaced by the set of all $(d-1)$-dimensional hypersurfaces in M, cf. ref. [4]. However, if $d > 2$ one has to assume that G is abelian.

We use a method presented in [4] to construct an S^1-valued stochastic cosurface. This cosurface will be a generalization of the Wilson loop $\exp(i \oint_C A(x) dx)$ in the sense that we also admit curves C that are not necessarily closed and that are not necessarily compact.

If we have a simple closed curve C with interior B, the law of $\exp(i \oint_C A(x)\, dx)$ is $Q_{1,\lambda^2(B)}$, where $Q_{1,t}, t \geq 0$, is a continuous convolution semigroup on S^1, cf. eq. (18). We assume that the probability measures $Q_{1,t}, t \geq 0$, have densities q_t with respect to the normalized Haar measure dx on S^1.

Let us take two curves that are piecewise C^1 and that have only finitely many self-intersections. We face the problem that these two curves may intersect infinitely many times. To exclude such pathologies, let us first consider straight lines connecting points of a lattice $\varepsilon\mathbb{Z}^2, \varepsilon > 0$. For brevity we shall call such straight lines "lattice curves". We also admit lattice curves that are not compact and that are not closed, but we do assume that there are only finitely many points where the lattice curves are not differentiable. The set of all such lattice curves will be denoted by $\mathcal{C}_\varepsilon$.

Take $(C_1, \ldots, C_n) \in \mathcal{C}_\varepsilon^n$. $(C_1, \ldots, C_n)$ is called "complex" in ref. [4]. If $C_{i_1} \cap C_{i_2} \subseteq \varepsilon\mathbb{Z}^2$ for $i_1 \neq i_2$ and if the complement of $\bigcup_{i=1}^n C_i$ is of the form $\mathbb{R}^2 \setminus \bigcup_{i=1}^n C_i = \bigcup_{j=1}^k B_j$, where the B_j are simply connected, $(C_1, \ldots, C_n)$ is called "regular saturated complex". In this case the distribution of $(K(C_1), \ldots, K(C_n))$ is defined by

$$\mathrm{d}P_{(K(C_1),\ldots,K(C_n))} = \left(\prod_{j=1}^k q_{\lambda^2(B_j)} \left(\prod_{C_i \subseteq \partial B_j} x_i^{\sigma_{ij}} \right) \right) \mathrm{d}x_1 \otimes \cdots \otimes \mathrm{d}x_n \,, \tag{28}$$

where $\sigma_{ij} = 1$ if C_i and ∂B_j have the same orientation and -1 otherwise. We put $q_\infty \equiv 1$.

If we have an arbitrary complex $(C_1, \ldots, C_n)$ we can find a regular saturated complex $(\tilde{C}_1, \ldots, \tilde{C}_m)$, $m > n$, containing $(C_1, \ldots, C_n)$, i.e., $\forall C_i \in \{C_1, \ldots, C_n\}$ $\exists \tilde{C}_{l_1}, \ldots, \tilde{C}_{l_r} \in \{\tilde{C}_1, \ldots, \tilde{C}_m\}$ such that $C_i = \tilde{C}_{l_1} \cup \cdots \cup \tilde{C}_{l_r}$. We define the distribution of $(K(C_1), \ldots, K(C_n))$ in a natural way by employing the property $K(C_1 \cup C_2) = K(C_1) \cdot K(C_2)$. Thus we get a projective system of probability measures. The projective limit is a probability measure on $(\mathrm{S}^1)^{\mathcal{C}_\varepsilon}$. Employing the continuity of the convolution semigroup $Q_{1,t}, t \geq 0$, we get the following theorem.

Theorem 5.2. *Let C be a curve, not necessarily connecting lattice points, that is closed, piecewise C^1, and that has only finitely many self-intersections. We can construct a sequence $K_{\varepsilon_n}, n \in \mathbb{N}$, of lattice cosurfaces and a sequence of lattice curves $C_{\varepsilon_n}, n \in \mathbb{N}$, $C_{\varepsilon_n} \in \mathcal{C}_{\varepsilon_n}$ $\forall n$, such that $C_{\varepsilon_n} \overset{n\to\infty}{\longrightarrow} C$ in the sense of (24) and $K_{\varepsilon_n}(C_{\varepsilon_n}) \overset{n\to\infty}{\longrightarrow} \exp(\mathrm{i} \oint_C A(x)\,\mathrm{d}x)$ weakly.*

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